R. Vaidyanathaswami
th the author's best compliments
B. S. Madhava Rao

Reprinted from THE MATHEMATICS STUDENT, *Vol. I, No. 3.*

Note on a Theorem on Homologous Stars

BY

B. S. MADHAVA RAO.

NOTE ON A THEOREM ON HOMOLOGOUS STARS.

BY B. S. MADHAVA RAO,
College of Engineering, Bangalore.

1. This note is concerned with the proof of the well-known theorem that

> ***the solutions of Emden's polytropic equation obtained by assuming different central conditions fall into a homologous series.****

The usual proofs assume the series of configurations to be of the same mass, and for such polytropic stars, it is shown that the ratio $\bar{\rho}/\rho_c$ is independent of the central conditions whence it follows that all these stars have the same law of density-distribution.† It has, however, been pointed out by Milne‡ that the concept of "***homologous stars series***" should be generalised so as to include stars of all masses and all radii having the same density-distribution thus leading to a doubly infinite system which might be called a "***homologous family***"

---

* Jeans, *Astronomy, and Cosmogony* (1928), p. 71.

† Emden Thermo-dynamik der Himmelskorper (*Ency. Math. Wiss.* (Bd. VI, 2B), p. 408, § 19.

‡ M. N. R. A. S., Nov. 1929, p. 27, § 14.

According to this definition, it is quite easy to see that the solutions of Emden's equation (even when we restrict ourselves to the particular Emden-Eddington solutions without taking into account the generalised Milne-Fowler solutions) with different central conditions form such a family and it is the object of this note to establish that this family is homologous.

2. A homologous series is defined usually to be a series of configurations of constant mass which satisfy Lane's laws, *viz.*, that if mass of gas be in equilibrium with the values r, ρ, P, T, these values can be changed to r', ρ', P$'$, T$'$ given by

$$r' = \alpha r; \quad \rho' = \alpha^{-3}\rho; \quad P' = \alpha^{-4}P; \quad T' = \alpha^{-1}T$$

and the mass will still be in equilibrium.* In order to prove the theorem in question for the generalised family it is therefore only necessary to use Lane's laws in a form suitable for application to the case of variable mass. Such a form is well-known † and is to the effect that at corresponding points within any number of stars of a homologous family (***corresponding points*** being those points whose distances from the centres are the same fractions of the radii) the densities vary as M/r^3, the pressures as M^2/r^4 and the temperatures as $\mu M/r$, where μ is the molecular weight at the point considered.

3. We now proceed with the proof adopting throughout the notation employed by Eddington.‡

The fundamental equation of polytropic gas spheres are

$$dP = -g\rho \, dr \qquad \dots \quad (1)$$

and

$$\frac{d^2\phi}{dr^2} + \frac{2}{r}\frac{d\phi}{dr} = -4\pi G\rho \qquad \dots \quad (2)$$

---

* Jeans. *Astronomy and Cosmogony*, p. 65.

† Russell-Dugan-Stewart. *Astronomy*, Vol. 2. p. 884.

‡ *Internal Constitution of Stars*, Chap. IV.

where ϕ is the gravitational potential, and the polytropic equation,

$$P = k\rho^{\gamma} \qquad \ldots \quad (3)$$

where $\gamma = 1 + 1/n$, n being the polytropic index. From equations (2) and (3) we easily deduce the equation

$$\frac{d^2\phi}{dr^2} + \frac{2}{r}\frac{d\phi}{dr} + \alpha^2\phi^n = 0 \qquad \ldots \quad (4)$$

where $\alpha = 4\pi G / \{ (n+1)^k \}^n$

with the further relations,

$$\rho = \left\{ \frac{\phi}{(n+1)k} \right\}^n \qquad \ldots \quad (5)$$

and $$P = \frac{\rho\phi}{n+1} \qquad \ldots \quad (6)$$

If ϕ_0 be the value of ϕ at the centre, we then substitute

$$\phi = \phi_0 u; \quad r = z / \alpha\phi_0^{\frac{1}{2}(n-1)} \qquad \ldots \quad (7)$$

in equation (4) and deduce Emden's equation in the standard form

$$\frac{d^2u}{dz^2} + \frac{2}{z}\frac{du}{dz} + u^n = 0 \qquad \ldots \quad (8)$$

with the central conditions, $u = 1$ and $du/dz = 0$ when $z = 0$.

If we have to consider, now, a solution of Emden's equation (8) with a different central condition, $u = u_0$ and $du/dz = 0$, when $z = 0$ we need only alter the substitutions (7) replacing them by

$$\phi/\phi_0 = u/u_0 \text{ and } r = z/\alpha(\phi_0/u_0)^{\frac{1}{2}(n-1)} \qquad \ldots \quad (9)$$

which would again transform (4) to the same equation (8), but with different central conditions. Let ρ', P', T', r', M' refer to the polytropic sphere with the central condition $u = u_0$ and the same quantities without accents for $u = 1$. We then have

$$\rho = \left\{ \frac{\phi_0 u}{(n+1)^k} \right\}^n ; \quad \rho' = \left\{ \frac{(\phi_0 / u_0) u}{(n+1) k} \right\}^n \qquad \text{from (5)}$$

$$P = \frac{\rho \phi}{n+1} \quad ; \quad P' = \frac{\rho' \phi'}{n+1} \qquad \text{from (6)}$$

where ϕ' stands for $(\phi_0 / u_0) u$.

As regards temperature the assumption of the perfect gas equation, $p_G = \frac{\mathscr{R}}{\mu} \rho T$ leads to the relation $T = \frac{\beta \mu \phi}{(n+1) \mathscr{R}}$, where β is the ratio of gas pressure to the whole pressure and $\mathscr{R}$ the universal gas constant We thus have

$$T = \frac{\beta \mu \phi}{(n+1) \mathscr{R}} ; \quad T' = \frac{\beta \mu \phi'}{(n+1) \mathscr{R}}$$

(assuming β and μ to be the same in both the cases).

Hence $\rho / \rho' = u_0^n$; $P/P' = u_0^{n+1}$; $T/T' = u_0$.

Again, $GM = \left(- r^2 \frac{d\phi}{dr} \right)_{u=0} = \frac{1}{a \phi_0^{\frac{1}{2}(n-3)}} \left(- z^2 \frac{du}{dz} \right)_{u=}$

and, $GM' = \frac{1}{a (\phi_0 / u_0)^{\frac{1}{2}(n-3)}} \left(- z^2 \frac{du}{dz} \right)_{u=0}$

$\therefore$ $$M/M' = u_0^{\frac{1}{2}(n-3)}$$

and finally $$r'/r = u_0^{\frac{1}{2}(n-1)}.$$

Hence, $\left(\frac{M}{r^3} \right) / \left(\frac{M'}{r'^3} \right) = \left(\frac{M}{M'} \right) \left(\frac{r'}{r} \right)^3 = u_0^{-\frac{1}{2}(n-3)} \cdot u_0^{\frac{3}{2}(n-1)}$

$$= u_0^n = \left(\frac{\rho}{\rho'} \right).$$

Again $\left(\frac{M^2}{\gamma^4}\right) \Big/ \left(\frac{M'^2}{\gamma'^4}\right) = \left(\frac{M}{M'}\right)^2 \left(\frac{r'}{r}\right)^4 = u_o^{-(n-3)} \cdot u_o^{2(n-1)}$

$$= u_o^{n+1} = \left(\frac{P}{P'}\right)$$

and finally, $\left(\frac{M}{r}\right) \Big/ \left(\frac{M'}{r'}\right) = u^{-\frac{1}{2}(n-3)} \cdot u_o^{\frac{1}{2}(n-1)} = u_o = \left(\frac{T}{T'}\right)$

Thus from the generalised form of Lane's laws, it follows immediately that the solutions form a homologous family. We can also easily verify that the ratio $\bar{\rho}/\rho_c$ is the same for all members of the family, for,

$$\bar{\rho} = M / \tfrac{4}{3}\pi R^3 = \frac{1}{G}\left(-\frac{1}{4/3\,\pi r^3} \cdot r^2 \frac{d\phi}{dr}\right)_{u=o}$$

$$= \frac{a^2 \phi_o^n}{4\pi G}\left(-\frac{3}{z} \cdot \frac{du}{dz}\right)_{u-o}$$

and, $\qquad a^2 \phi_o^n = 4\pi G \rho_c.$

Thus $\qquad (\bar{\rho}/\bar{\rho}') = u_o^n$ and $(\rho_c/\rho_c') = u_o^n$

which shows $\qquad (\bar{\rho}/\rho_c) = (\bar{\rho}'/\rho_o').$

S. V. & CO., 190, MOUNT ROAD—MADRAS.